ON THE RELATIVISTIC OSCILLATORY MOTION

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ABSTRACT

The relativistic oscillatory motion is studied by the method of canonical transformation. Two types of relativistic oscillators are considered: (1) relativistic harmonic oscillator, and (2) relativistic damped oscillator. The transformations are performed in a mathematical way according to an operational method of relativistic perturbations conserving Hamiltonian formulism. The ultimate nature of the relativistic oscillation is analyzed.

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I. INTRODUCTION

Struble and Harris[1] have criticized Mitchell and Pope's[2] work on the relativistic damped oscillator and have presented their results on the same problem. However, the solution given by Struble and Harris does not permit a deeper insight into the nature of the problem because it is assumed in a series form.

In this paper canonical transformations are offered for the analysis of the relativistic oscillatory motion. The advantage of the use of canonical system of equations to treat this problem is that solutions can be obtained in a certain desired form so that the ultimate nature of the relativistic oscillatory motion is revealed. The method of canonical transformation, an adaptation of the principle developed originally by Von Zeipel[3], confirms the results obtained by Penfield and Zatskis[4], Mitchell and Pope[2]. The Von Zeipel's method has been used with success in the study of motion of artificial satellites[5]. Its application to this type of problem is apparently new.

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II. RELATIVISTIC EQUATIONS OF MOTION

The three-dimensional form of Einstein's second law of motion for a particle is [6]

$$d\mathbf{p}/dt = \mathbf{F}; \quad \mathbf{p} = m_0/(1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}\,\mathbf{v} \tag{1}$$

in which m_0 is the rest mass, $\mathbf{v}$ the velocity of the particle, C the velocity of light in a vacuum and $\mathbf{F}$ represents force.

If the motion of the particle is described relative to the inertial coordinate system in which the origin coincides with the equilibrium position of the particle and the X-axis represents the straight line on which it moves, the equations of oscillatory motion are

$$dp/dt = -(kx + r\dot{x}); \quad p = m_0/(1 - \dot{x}^2/c^2)^{\frac{1}{2}}\,\dot{x} \tag{2}$$

where k is the spring modulus and r the damping constant. The initial conditions assumed to be $x = a$ and $\dot{x} = 0$ at time $t = 0$. If one introduces the Newtonian angular frequency $\omega = (k/m_0)^{\frac{1}{2}}$, the Newtonian damping coefficient $\eta = r/2m_0$ and the characteristic length a, equation (2) can be nondimensionalized by changing variable to $\xi = x/a$, a nondimensional space variable and to $\tau = \omega t$, nondimensional time. Using $p = d\xi/d\tau$ as the momentum

(6) P. Havas, Rev. Mod. Phys. 36, 938 (1964)

conjugates to q equation (2) becomes

$$\frac{dp}{d\tau} + (q + 2\alpha p)(1 - \mu p^2)^{\frac{3}{2}} = 0 \tag{3}$$

where $\alpha = \gamma/\omega$ and $\mu = (a\omega/c)^2$ are independent small parameters. In terms of the nondimensional variables the initial conditions are q = 1 and p = 0 at τ = 0. Subjecting to these initial conditions a solution of equation (3) is desired.

III. CANONICAL VARIABLES AND TRANSFORMATIONS

Consider the dynamical system to be perturbed by external forces due to relativistic effect such that the canonical equations describing the motion of the system are

$$\frac{dq}{d\tau} = \frac{\partial H}{\partial p} + X(p,q,\tau) \tag{4}$$

$$\frac{dp}{d\tau} = -\frac{\partial H}{\partial q} - Y(p,q,\tau) \tag{5}$$

where H = H(p,q) is the Hamiltonian of the system, and $X(p,q,\tau)$ and $Y(p,q,\tau)$ are external perturbing forces due to the relativistic effect.

A fundamental theorem in the transformation theory of theoretical mechanics is that a transformation of variables conserving the Hamiltonian formulism may be made providing the Hamiltonian does not have an explicit dependence on time. If an appropriate transformation method is used, the equations describing the motion of the system will remain canonical although of a different functional form of the Hamiltonian.

Let a transformation to new variables be given in the form

$$H(p,q) = H'(L,l) \tag{6}$$

The question is, under what circumstances will the transformation be canonical--i.e., under what circumstances the new equations will be

$$\frac{dl}{d\tau} = \frac{\partial H'}{\partial L} + F(L, l, \tau) \tag{7}$$

$$\frac{dL}{d\tau} = -\frac{\partial H'}{\partial l} - G(L, l, \tau) \tag{8}$$

This transformation of variables can be done in the following manner. Multiply equations (4) and (5) by $\partial p/\partial L$, $-\partial q/\partial L$ respectively and add. The sum is

$$\frac{dq}{d\tau}\cdot\frac{\partial p}{\partial L} - \frac{dp}{d\tau}\cdot\frac{\partial q}{\partial L} = \frac{\partial H}{\partial p}\frac{\partial p}{\partial L} + \frac{\partial H}{\partial q}\frac{\partial q}{\partial L} + X\frac{\partial p}{\partial L} + Y\frac{\partial q}{\partial L} \tag{9}$$

Since

$$\frac{dq}{d\tau} = \frac{\partial q}{\partial l}\frac{dl}{d\tau} + \frac{\partial q}{\partial L}\frac{dL}{d\tau}$$

$$\frac{dp}{d\tau} = \frac{\partial p}{\partial l}\frac{dl}{d\tau} + \frac{\partial p}{\partial L}\frac{dL}{d\tau}$$

and

$$\frac{\partial H'}{\partial L} = \frac{\partial H}{\partial p}\frac{\partial p}{\partial L} + \frac{\partial H}{\partial q}\frac{\partial q}{\partial L}$$

equation (9) becomes

$$\left[\frac{\partial q}{\partial l}\frac{\partial p}{\partial L} - \frac{\partial p}{\partial l}\frac{\partial q}{\partial L}\right]\frac{dl}{d\tau} = \frac{\partial H'}{\partial L} + X\frac{\partial p}{\partial L} + Y\frac{\partial q}{\partial L} \tag{10}$$

It can be shown that the Lagrangian bracket $[l, L]$ equals unity for a canonical transformation where the Hamiltonian is unchanged. Hence

$$\frac{dl}{d\tau} = \frac{\partial H'}{\partial L} + X\frac{\partial p}{\partial L} + Y\frac{\partial q}{\partial L} \tag{11}$$

Comparing equation (11) with equation (7) one obtains

$$F = X\frac{\partial p}{\partial L} + Y\frac{\partial q}{\partial L} \tag{12}$$

If equations (4) and (5) are multiplied by $\partial p/\partial l$, $-\partial q/\partial l$ respectively and similar operations are carried out the result is

$$G = X\frac{\partial p}{\partial l} + Y\frac{\partial q}{\partial l} \tag{13}$$

IV. RELATIVISTIC HARMONIC OSCILLATOR

The case of the relativistic oscillator without damping is a relativistic harmonic oscillator. One obtains the equation of motion by placing $\alpha = 0$ in equation (3)

$$\frac{dp}{d\tau} + q - \frac{3}{2}\mu p^2 q = 0 \tag{14}$$

where terms of order μ^2 and higher are omitted. The appearances of μ is a direct consequence of the relativistic perturbation of the problem. The Hamiltonian of the Newtonian harmonic oscillator is

$$H_0 = \frac{1}{2}(p^2 + q^2) \tag{15}$$

and the solution of the Newtonian harmonic oscillator subjecting to the initial conditions $p = 0$ and $q = q_0$ can be expressed in the periodic form

$$\begin{aligned} q &= q_0 \cos(\tau - \tau_0) \\ p &= -q_0 \sin(\tau - \tau_0) \end{aligned} \tag{16}$$

By letting $l = \tau - \tau_0$ and the total energy of the Newtonian system $L = \frac{1}{2} q_0^2$ so that equations (16) are transformed to

$$q = \sqrt{2L} \cos l \tag{17}$$

$$p = -\sqrt{2L} \sin l \tag{18}$$

and the canonical equations of the relativistic system are

$$\frac{dq}{d\tau} = \frac{\partial H_0}{\partial p} \tag{19}$$

$$\frac{dp}{d\tau} = -\frac{\partial H_0}{\partial q} + \frac{3}{2}\mu p^2 q \tag{20}$$

where

$$H_0 = L = H'(L,-) \tag{21}$$

Upon differentiating equations (17) and (18), the results are

$$\frac{\partial q}{\partial l} = p \quad ; \quad \frac{\partial q}{\partial L} = \frac{\cos^2 l}{q}$$

By substituting these results equations (12) and (13) become

$$F = -\frac{3}{2}\mu p^2 \cos^2 l = -\frac{3}{16}\mu q_0^2 + \frac{3}{16}\mu q_0^2 \cos 4l \tag{22}$$

$$G = -\frac{3}{2}\mu p^3 q = -\frac{3}{16}\mu q_0^4 (4\sin 2l + \sin 4l) \tag{23}$$

Substituting equations (21), (22) and (23) in equations (7) and (8) one obtains

$$\frac{dl}{d\tau} = 1 - \frac{3}{16}\mu q_0^2 + \frac{3}{16}\mu q_0^2 \cos 4l \tag{24}$$

$$\frac{dL}{d\tau} = -\frac{3}{16}\mu q_0^4 (4\sin 2l + \sin 4l) \tag{25}$$

Therefore the solution of equation (14) is

$$q = \sqrt{2L}\,\cos l \tag{26}$$

where L and l are given by equations (24) and (25).

The physical interpretation of these results is as follows. The displacement after an interval $\tau - \tau_0$. may be determined from the Newtonian harmonic oscillator by considering that the amplitude and frequency are time varying due to the relativistic perturbation. The variation in frequency is inherent in equation (24). The variation in amplitude is given by equation (25). The results also shows that the amplitude of the relativistic harmonic oscillatory motion to have no secular terms. It does not increase or decrease without bound but is represented by a constant plus trigonometric terms having frequencies which are multiples of the fundamental frequency of the system.

Taking the values averaged over one period, one obtains

$$\left.\frac{dl}{d\tau}\right|_{AVE} = 1 - \frac{3}{16}\mu q_0^2$$

$$\left.\frac{dL}{d\tau}\right|_{AVE} = 0$$

Substituting these average values in equation (26) in order to obtain a first approximation to the solution, the result is

$$q = q_0 \cos\left[\left(1 - \frac{3}{16}\mu q_0^2\right)\tau - \tau_0\right] \tag{27}$$

Thus, referring to the definitions of q, q_0 and τ, the first approximation to the solution of the relativistic harmonic oscillator is

$$x_{r.h.o.} = a\cos\left[\left(1-\tfrac{3}{16}\mu\right)\omega t\right] \tag{28}$$

Equation (28) represents the relativistic harmonic oscillation with constant amplitude and constant frequency. The frequency is $\frac{3}{32\pi}\mu\omega$ less than that of the Newtonian harmonic oscillator. Penfield and Zatzkis[4] arrive at the same result by using the energy principle in relativistic dynamics.

V. RELATIVISTIC DAMPED OSCILLATOR

Equation (3) describes the motion of the relativistic damped oscillator. Further transformation of this equation procedes the application of the canonical transformation. After suitable changes of both independent and dependent variables, and a reidentification of the parameters, the equation of motion assumes a form particularly suited for the Von Zeipel's technique.

Equation (3) is rewritten

$$\frac{dp}{d\tau} + 2\alpha p + q - \frac{3}{2}\mu(p^2 q + 2\alpha p^3) + \frac{3}{8}\mu^2(p^4 q + 2\alpha p^5) + O(\mu^3) = 0 \tag{29}$$

Changing of the independent and dependent variables according to

$$q = q_1 e^{-\alpha\tau}\ ; \quad \tau = (1-\alpha^2)^{-\frac{1}{2}}T$$

TABLE II - CONTINUED

Transition	R_{AB}	1st Order Dipole Length	1st Order Dipole Velocity	2nd Order Dipole Length	2nd Order Dipole Velocity	Exact Value
$3p\sigma$-$3d\sigma$						
	0.00					1.732
	0.25			1.732	1.734	
	0.50			1.726	1.730	
	0.75			1.705	1.710	
	1.00			1.667	1.671	
	2.00			1.314	1.357	
	3.00			0.821	0.786	
	4.00			0.347	0.217	
$3p\sigma$-$3s\sigma$						
	0.00					2.452
	0.25	2.399	2.387	2.403	2.394	
	0.50	2.221	2.224	2.235	2.229	
	0.75	1.978	1.988	2.001	1.994	
	1.00	1.761	1.771	1.789	1.780	
	2.00	1.395	1.362	1.450	1.462	
	3.00	1.412	1.306	1.649	1.771	
	4.00	1.559	1.358	2.116	2.491	
$3d\sigma$-$3s\sigma$						
	0.00					0.000
	0.25			0.058	0.059	
	0.50			0.106	0.107	
	0.75			0.134	0.136	
	1.00			0.137	0.141	
	2.00			0.019	0.033	
	3.00			-0.350	-0.347	
	4.00			-0.796	-0.843	

and identifying the new parameters

$$\epsilon = \mu(1-\alpha^2)\,; \quad \Lambda = \alpha(1-\alpha^2)^{-\frac{1}{2}}$$

enables one to write equation (29) in the form

$$\frac{dp_1}{dT} + q_1 - \frac{3}{2}\epsilon(p_1-\Lambda q_1)^2\left[(1-\Lambda^2)q_1 + 2\Lambda p_1\right]e^{-2\Lambda T}$$

$$+ \frac{3}{8}\epsilon^2(p_1-\Lambda q_1)^4\left[(1-\Lambda^2)q_1 + 2\Lambda p_1\right]e^{-4\Lambda T}$$

$$+ O(\epsilon^3) = 0 \tag{30}$$

The new parameter ϵ couples the independent parameters μ and α. The restriction $\alpha < 1$ ensures that ϵ will be smaller than μ. It is to be noted that the appearance of ϵ is a direct consequence of the relativistic perturbation of the system. The Hamiltonian of the unperturbed system is

$$H_{01} = \frac{1}{2}(p_1^2 + q_1^2) \tag{31}$$

and the solution of the unperturbed system subjecting to the transformed initial conditions $q_1(0) = q_{01}$ and $p_1(0) = \Lambda q_{01}$ can be expressed in the periodic form

$$q_1 = q_{01} \cos(T - T_0) + \Delta q_{01} \sin(T - T_0) \tag{32}$$

$$p_1 = -q_{01} \sin(T - T_0) + \Delta q_{01} \cos(T - T_0) \tag{33}$$

By letting $l = T - T_0$ and the total energy $L = \frac{1}{2}(1+\Delta^2) q_{01}^2$ so that equations (32) and (33) are transformed to

$$q_1 = \sqrt{\frac{2L}{1+\Delta^2}} \cos l + \Delta \sqrt{\frac{2L}{1+\Delta^2}} \sin l \tag{34}$$

$$p_1 = -\sqrt{\frac{2L}{1+\Delta^2}} \sin l + \Delta \sqrt{\frac{2L}{1+\Delta^2}} \cos l \tag{35}$$

The canonical equations of the perturbed system are

$$\frac{dq_1}{dT} = \frac{\partial H_{01}}{\partial p_1} \tag{36}$$

$$\frac{dp_1}{dT} = -\frac{\partial H_{01}}{\partial q_1} + \frac{3}{2}\epsilon\left[2\Delta p_1^3 + (1-5\Delta^2) p_1^2 q_1 + (4\Delta^3 - 2\Delta) p_1 q_1^2 + (1-\Delta^2)\Delta^2 q_1^3\right] \tag{37}$$

where

$$H_{01} = L = H'(L, -) \tag{38}$$

In this expression terms of order ϵ^2 and higher are omitted.

By differentiation (34) and (35) the results are

$$\frac{\partial q_1}{\partial l} = p_1$$

$$\frac{\partial q_1}{\partial L} = \frac{(\cos l + \Delta \sin l)^2}{(1+\Delta^2) q_1}$$

Substituting these results in equations (12) and (13), one obtains

$$F = -\frac{3}{2}\epsilon\left[2\Delta p_1^3 + (1-5\Delta^2)p_1^2 q_1 + (4\Delta^3 - 2\Delta)p_1 q_1^2 + (1-\Delta^2)\Delta^2 q_1^3\right] \cdot \frac{(\cos l + \Delta \sin l)^2}{(1+\Delta^2) q_1} \cdot e^{-2\Delta T}$$

$$= -\frac{3}{2}\epsilon\Big[2\Delta(-\sin l + \Delta \cos l)^3 + (1-5\Delta^2)(-\sin l + \Delta \cos l)^2 \cdot (\cos l + \Delta \sin l) + (4\Delta^3 - 2\Delta)(-\sin l + \Delta \cos l)(\cos l + \Delta \sin l)^2 + (1-\Delta^2)\Delta^2(\cos l + \Delta \sin l)^3\Big] \frac{(\cos l + \Delta \sin l)}{1+\Delta^2} q_{01}^2 e^{-2\Delta T} \qquad (39)$$

$$G = -\frac{3}{2}\epsilon\left[2\Delta p_1^4 + (1-5\Delta^2)p_1^3 q_1 + (4\Delta^3 - 2\Delta)p_1^2 q_1^2 + (1-\Delta^2)\Delta^2 p_1 q_1^3\right] e^{-2\Delta T}$$

$$= -\frac{3}{2}\epsilon\left[2\Delta(-\sin l + \Delta\cos l)^4 + (1-5\Delta^2)(-\sin l + \Delta\cos l)^3(\cos l + \Delta\sin l)\right.$$

$$+ (4\Delta^3 - 2\Delta)(-\sin l + \Delta\cos l)^2(\cos l + \Delta\sin l)^2 + (1-\Delta^2)\Delta^2$$

$$\left.(-\sin l + \Delta\cos l)(\cos + \Delta\sin l)^3\right]\cdot q_{01}^4 \cdot e^{-2\Delta T} \tag{40}$$

Equations (39) and (40) can be expressed as the sum of two terms, one secular and the other periodic in l. By expressing

$$\sin^4 l = \frac{3}{8} - \frac{1}{2}\cos 2l + \frac{1}{8}\cos 4l$$

$$\cos l \sin^3 l = \frac{1}{4}\sin 2l - \frac{1}{8}\sin 4l$$

$$\cos^2 l \sin^2 l = \frac{1}{8} - \frac{1}{8}\cos 4l$$

$$\cos^3 l \sin l = \frac{1}{4}\sin 2l + \frac{1}{8}\sin 4l$$

$$\cos^4 l = \frac{3}{8} + \frac{1}{2}\cos 2l + \frac{1}{8}\cos 4l$$

equations (39) and (40) become

$$F = -\tfrac{3}{2}\epsilon\left(f_0 + f_1 \sin 2l + f_2 \cos 2l + f_3 \sin 4l + f_4 \cos 4l\right) q_{01}^2 e^{-2\Lambda T} \tag{41}$$

$$G = -\tfrac{3}{2}\epsilon\left(g_0 + g_1 \sin 2l + g_2 \cos 2l + g_3 \sin 4l + g_4 \cos 4l\right) q_{01}^4 e^{-2\Lambda T} \tag{42}$$

where

$$
\begin{aligned}
f_0 &= \tfrac{1}{8}\left(1-\Lambda^2-5\Lambda^4-3\Lambda^6\right)\\
f_1 &= -\Lambda\left(1-\Lambda-2\Lambda^2+\Lambda^3+\Lambda^4\right)\\
f_2 &= \tfrac{1}{2}\Lambda^2\left(3-6\Lambda^2-\Lambda^4\right)\\
f_3 &= \frac{\Lambda}{2(1+\Lambda^2)}\left(1-\Lambda-7\Lambda^2+6\Lambda^3+7\Lambda^4-\Lambda^5-\Lambda^6\right)\\
f_4 &= \frac{1}{8(1+\Lambda^2)}\left(1-12\Lambda^2+28\Lambda^4-8\Lambda^6+\Lambda^8\right)
\end{aligned}
$$

$$
\begin{aligned}
g_0 &= \tfrac{1}{2}\Lambda\left(1+\Lambda+2\Lambda^2+2\Lambda^3+\Lambda^4+\Lambda^5\right)\\
g_1 &= \tfrac{1}{4}\left(1-12\Lambda^2+6\Lambda^4+4\Lambda^6+\Lambda^8\right)\\
g_2 &= -\tfrac{1}{2}\left(2-\Lambda+5\Lambda^3-2\Lambda^4+5\Lambda^5-\Lambda^7\right)\\
g_3 &= \tfrac{1}{8}\left(1+\Lambda+5\Lambda^2-6\Lambda^3-\Lambda^4+\Lambda^5+17\Lambda^6-\Lambda^8\right)\\
g_4 &= -\tfrac{1}{8}\Lambda^2\left(24-16\Lambda-28\Lambda^2+17\Lambda^3+4\Lambda^4-\Lambda^5\right)
\end{aligned}
$$

Substituting equations (38), (41) and (42) in equations (7) and (8) one obtains

$$\frac{dl}{dT} = 1 - \frac{3}{2}\epsilon\left(f_0 + f_1 \sin 2l + f_2 \cos 2l + f_3 \sin 4l + f_4 \cos 4l\right) q_{01}^2 e^{-2\Delta T} \tag{43}$$

$$\frac{dL}{dT} = \frac{3}{2}\epsilon\left(g_0 + g_1 \sin 2l + g_2 \cos 2l + g_3 \sin 4l + g_4 \cos 4l\right) q_{01}^4 e^{-2\Delta T} \tag{44}$$

Therefore the solution of equation (30) is

$$q_1 = \sqrt{\frac{2L}{1+\Delta^2}} \cos l + \Delta \sqrt{\frac{2L}{1+\Delta^2}} \sin l \tag{45}$$

where L and l are given by equations (43) and (44).

Taking the values averaged over one period the results are

$$\left.\frac{dl}{dT}\right|_{AVE} = 1 - \frac{3}{16}\epsilon\left(1 - \Delta^2 - 5\Delta^4 - 3\Delta^6\right) e^{-2\Delta T} \cdot q_{01}^2$$

$$\left.\frac{dL}{dT}\right|_{AVE} = \frac{3}{4}\epsilon\Delta\left(1 + \Delta + 2\Delta^2 + 3\Delta^3 + \Delta^4 + \Delta^5\right) e^{-2\Delta T} \cdot q_{01}^4$$

Using these average values in order to obtain a first approximation to the solution one obtains

$$\frac{dq_{01}}{dT} = \frac{3}{4}\epsilon\Delta(1+\Delta+\Delta^2+\Delta^3)\cdot e^{-2\Delta T} q_{01}^3$$

Performing the integration the result is

$$q_{01}^* = \frac{q_{01}}{\left[1-\frac{3}{4}\epsilon\Delta(1+\Delta+\Delta^2+\Delta^3)(1-e^{-2\Delta T})q_{01}^2\right]^{\frac{1}{2}}} \tag{46}$$

Therefore the first approximation of q_1, is

$$q_1 = q_{01}^* \cos\left[T - \frac{3}{32\Delta}\epsilon(1-\Delta^2-5\Delta^4-3\Delta^6)(1-e^{-2\Delta T})q_{01}^2\right]$$

$$+\Delta q_{01}^* \sin\left[T - \frac{3}{32\Delta}\epsilon(1-\Delta^2-5\Delta^4-3\Delta^6)(1-e^{-2\Delta T})q_{01}^2\right] \tag{47}$$

Referring to the definitions of q, q_1, q_{01}, τ and T, the first approximate solution of the relativistic damped oscillator is

$$x_{r.d.o.} = e^{-\alpha\tau}\left[\frac{a}{(1-A)^{\frac{1}{2}}}\cos\left[(1-\alpha^2)^{\frac{1}{2}}\tau - B\right] + \Delta\frac{a}{(1-A)^{\frac{1}{2}}}\sin\left[(1-\alpha^2)^{\frac{1}{2}}\tau - B\right]\right] \tag{48}$$

where

$$A = \frac{3}{4}\epsilon\Lambda(1+\Lambda+\Lambda^2+\Lambda^3)(1-e^{-2\Lambda T})$$

$$B = \frac{3}{32\Lambda}\epsilon(1-\Lambda^2-5\Lambda^4-3\Lambda^6)(1-e^{-2\Lambda T})$$

The motion as described by equation (48) is a relativistic damped oscillatory motion. Since the relativistic amplitude coefficient A is always positive the damping of the amplitude here is less effective than that of a corresponding Newtonian damped oscillator. The frequency is constant. It is to be noted that the relativistic phase angle shift is determined by the value of the relativistic phase coefficient B. The difference of phase angles between the relativistic and the Newtonian damped oscillators is determined by the value of $(1-\Lambda^2-5\Lambda^4-3\Lambda^6)$. If $1-\Lambda^2-5\Lambda^4-3\Lambda^6 = 0$, the phase angle of the relativistic oscillator equals that of the Newtonian oscillator. The phase angle of the relativistic oscillator is larger or smaller than that of the Newtonian one according to $1-\Lambda^2-5\Lambda^4-3\Lambda^6 < 0$ or $1-\Lambda^2-5\Lambda^4-3\Lambda^6 > 0$ respectively.

VI. VERIFICATION OF SOLUTIONS

The ultimate nature of the relativistic oscillatory motion is concisely revealed by equation (48). The relativistic effect upon the amplitude and the relativistic phase shift are clearly exhibited. The numerical calculations of equation (48) for $\mu = 0.1$, $\alpha = 0.25$ agree with those given by Mitchell and Pope[2] in the range where their solution is valid. It can be shown that the solution represented by equation (48) meets the limiting conditions imposed. In the absence of relativistic effect, i.e., $\epsilon = 0$, the relativistic amplitude coefficient A and relativistic phase coefficient B become zero. Accordingly, equation (48) reduces to

$$x_{n.d.o.} = e^{-\alpha\tau}\left[a\cos(1-\alpha^2)^{\frac{1}{2}}\tau + \Delta a\sin(1-\alpha^2)^{\frac{1}{2}}\tau\right] \tag{49}$$

which is known as the standard solution for Newtonian damped oscillator. It also reduces to the solution for the undamped relativistic harmonic oscillatory motion, equation (28), as α becomes vanishingly small. Or more concisely equations (48) and (49) can be expressed in the following forms

$$x_{r.d.o.} = e^{-\alpha\tau}\frac{a(1+\Delta^2)^{\frac{1}{2}}}{(1-A)^{\frac{1}{2}}}\cos\left[(1-\alpha^2)^{\frac{1}{2}}\tau - B - \Phi\right] \tag{50}$$

$$x_{n.d.o.} = e^{-\alpha\tau}a(1+\Delta^2)^{\frac{1}{2}}\cos\left[(1-\alpha^2)^{\frac{1}{2}}\tau - \Phi\right] \tag{51}$$

where

$$\Phi = \tan^{-1}\Delta$$

Such a compactness of representation permits a deeper insight into the nature of the problem.

This presentation also shows that the Newtonian oscillator can indeed be considered as a close analogy as well as mathematical limit of the relativistic oscillatory motion. Equation (48) satisfies the equation of motion (3) subject to the initial conditions $p = 0$, $q = 1$ when $t = 0$ to within a quantity of the order of ϵ^2. The solution may be carried out to any desired degree of accuracy in terms of higher orders of parameter ϵ by use of the same method.

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